MMP Learning Seminar

Week 58:

- Global - to - local.
- Upper bound for the volume.
- Birational boundedness.

Theorem (Adjunction): Let $I \subseteq [0,1]$ be a subset containing 1.

Let X proj of dimension n. & $V \subseteq X$ be an irreducible subvariety

with normalization $W \longrightarrow V$. Suppose (X, Δ) is a log pair & $\Delta' \geq 0$ $\mathbb{R}$-Cartier

with the following properties:

(1) the coefficients of Δ belong to I,

(2) (X, Δ) is klt, and

(3) there exists a unique non-klt place ν of $(X, \Delta + \Delta')$ with center V.

There exists Θ divisor on W with coeff in $\{a \mid 1-a \in LCT_{n-1}(D(I))\} \cup \{1\}$

such that

$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta)$$

is pseff.

Assume V is a general element of a family covering X.

$U \xrightarrow{\psi} W$ the normalization and Ψ the strict transform of Θ,

Then $K_U + \Psi \geq (K_X + \Delta)|_U$.

Theorem 3.5.5:

$\mathcal{B}_0 = \{ (X,\Delta) \mid X$ proj n-dim, (X,Δ) klt & $K_X+\Delta$ ample$\}$.

Assume we can control p, k & ℓ:

1) $V \longrightarrow B$ dominant family of subvarieties. if $b \in B$.

there exists $0 \le \Delta_b \sim_{\mathbb{Q}} (1-\delta)H$, for some $\delta \ge 0$. s.t

there is a unique non-klt place of $(X, \Delta+\Delta_b)$ whose center is Δ_b

where $H = k(K_X+\Delta)$ (k)

2) D on $W \longrightarrow V_b$ such that ϕ_D is birational & $\ell H|_W - D$ pseff. (ℓ)

3) $p\Delta$ integral or Δ has standard coeff. (p) meaning $1-\frac{1}{n}$.

Then $\phi_{mk(K_X+\Delta)}$ is birational for all $(X,\Delta) \in \mathcal{B}_0$

Theorem A_n: Local ACC in dim n.

$\downarrow +1$

Theorem B_n: $(X,\Delta)\in\mathfrak{D} \iff (X,\Delta)$ dim n projective klt, coeff $\Delta \subseteq I$, & $K_X+\Delta \equiv 0$.

Then $\mathrm{vol}(\Delta)$ is bounded above. (upper bound for the volume).

Theorem C_n: $(X,\Delta)\in\mathfrak{B} \iff (X,\Delta)$ dim n, projective, lc, coeff$(\Delta)\subseteq I$, $K_X+\Delta$ big.

+1 Then (X,Δ) is log birationally bounded. (birationally bounded)

Theorem D_n: Global ACC.

$K_X+\Delta\equiv 0$, lc proj of dim n & coeff Δ are in a DCC,

then actually they belong to a finite set.

5. Global-to-local: $D_{n-1} \Rightarrow A_n$.

Lemma 5.1: Fix integer n, $I \subseteq [0,1]$, $1 \in I$.

(X,Δ) lc of dim $n+1$, $\operatorname{coeff}(\Delta) \subseteq I$, $V \subseteq X$ non-klt center.

$\underline{c \in I}$ a coeff of a component $\underline{M}$ of Δ containing V.

We can find (S,Θ) proj of dim $\leq n$, $\operatorname{coeff}(\Theta) \subseteq D(I)$.

$K_S + \Theta \equiv 0$ and some comp of Θ has coeff of the form

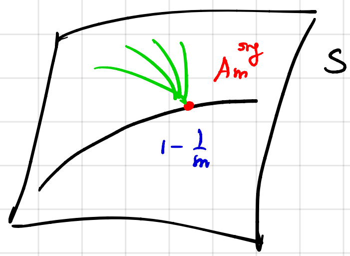
mult of M at the generic point of this prime comp.

$$\boxed{1 - \frac{1}{m}} + \boxed{\frac{f}{m}} + \boxed{\frac{kc}{m}}$$

orbifold sing contr.

contr of M

contr of comp which are not M



where $m, k \in \mathbb{N}$ and $\underline{f \in D(I)}$.

Proof: V is a unique lcc, every comp of Δ contain V.

Step 1: There is a component of Δ with coeff 1.

$Y \longrightarrow S \subseteq \Delta$, $(K_X+\Delta)|_Y = K_Y + \Gamma$, $\operatorname{coeff}(\Gamma) \in D(I)$

component N with coeff $d = \frac{\ell-1}{\ell} + \frac{g+jc}{\ell}$, containing the preimage of V.
$g \in I_+$.

inversion of adjunction, every component of the inverse image of V on Y is a lcc of (Y,Γ).

$\dim Y \leq n-1$ Apply induction.

$$D(D(I)) = D(I)$$

Step 2: The statement when $\lfloor \Delta \rfloor = 0$.

$f: Y \longrightarrow X$ dlt modification of (X, Δ). Y $\mathbb{Q}$-factorial.

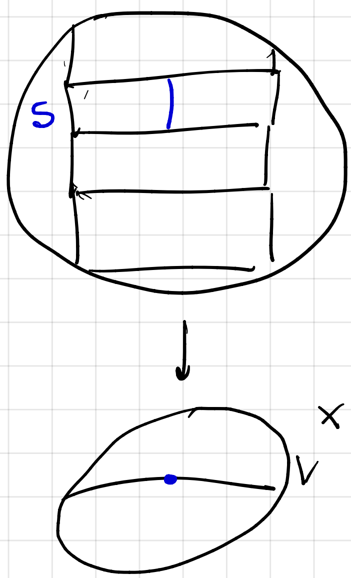
$K_Y + T + \Gamma = f^*(K_X + \Delta)$, T is the sum of exc.

We may that the inverse image of V is contained in T.

$S \subseteq T$ irr component that int the strict transform N of M

$$(K_Y + \Gamma + T)|_S = K_S + \Theta.$$

(S, Θ). have all the conditions except it may not be proj



The map $S \longrightarrow V$ is proj,

so we may replace S with

a general fiber of it $\square$

Lemma 5.2: $I \subseteq [0,1]$ DCC, $J_0 \subseteq [0,1]$ finite. Then

$$I_0 = \left\{ c \in I \;\middle|\; \frac{m - 1 + f + kc}{m} \in J_0 \text{ for some } k \in \mathbb{N} \ \& \ f \in D(I) \right\}$$

is finite.

Lemma 5.3: $D_{n-1} \Longrightarrow A_n$.

Proof: I DCC, so does $J = D(I)$.

Thm $D_{n-1} \Longrightarrow$ (S, Θ) dim $\leq n-1$, coeff $\Theta \subseteq J$,

then the coeff $\Theta \subseteq J_0$ for some $J_0 \subseteq J$.

Set $I_0 = \left\{ c \in I \;\middle|\; \frac{m-1+f+kc}{m} \in J_0 \text{ for some } k_0, m \in \mathbb{N},\ f \in I_+ \right\}$

J_0 is finite $\Longrightarrow$ I_0 is finite.

(X, Δ) is lc, coeff $(\Delta) \subseteq I$, $Z \subseteq X$ non-klt contained

in every single comp of Δ, $(5.1) \Longrightarrow$ coeff Δ belong to I_0 $\square$.

Upper bound for the volume:

$\mathcal{D} := \{ (X,\Delta)$ dim n, coeff $\Delta \subseteq I$, $K_X + \Delta \equiv 0$ $\}$ vol(Δ) is bounded above

Lemma: Assume D_{n-1} & A_{n-1}. There exists $\varepsilon > 0$ such that
If $(X,\Delta) \in \mathcal{D}$, Δ big, $K_X + \Phi \equiv 0$, where

$$\Phi \geq (1-\delta)\Delta.$$

for some $\delta < \varepsilon$, then (X,Φ) is klt.

Proof:

$A_{n-1} + D_{n-1}$

(S,Θ) of dim $n-1$, coeff $\Theta \in D(I)$ and

$(1-\varepsilon)\Theta \leq \Theta' \leq \Theta$ (ε only depending on $n-1$).

then (S,Θ) lc $\Longleftrightarrow$ (S,Θ') is lc.

Moreover $\Theta = \Theta'$ if $K_S + \Theta' \equiv 0$.

Step 1: To reduce to the case in which (X,Φ) is plt.

Step 2: Reduce to the case in which all divisors in the picture are $\mathbb{Q}$-proportional.

$\phi : Y \longrightarrow X$ dlt modification of (X,Φ)

$K_Y + \Psi = \phi^*(K_X + \Phi)$

$K_Y + \Gamma + \alpha S = \phi^*(K_X + \Delta)$ $\qquad\qquad \rho(W/Z) = 1$

$(-S)$-negative

$\alpha < 1$, $S = \lfloor \Psi \rfloor$. $\qquad K_Y + \Psi \equiv 0$

$K_Y + \Psi - S$ is not pseff. Run a $(K_Y + \Psi - S)$-MMP.

$$\begin{array}{c} Y \overset{p}{\dashrightarrow} W \\ \quad\downarrow \pi \\ \quad Z \end{array}$$

All the steps of the MMP are S-positive

the strict transform T of S in W is horizontal over the base Z.

F a general fiber of π, $(Y,\Gamma,\Psi) \longmapsto (F, p_*\Gamma|_F, p_*\Psi|_F)$

Step 3: We conclude using the first paragraph

$K_Y + \Gamma + S$ is ample. $\quad \psi \geq (1-\varepsilon)\Gamma + S$, then.

$K_Y + (1-\eta)\Gamma + S \equiv 0$ for some $0 < \eta < \varepsilon$, and.

$K_Y + (1-\varepsilon)\Gamma + S$ is log canonical.

$$(K_Y + (1-\varepsilon)\Gamma + S)|_S = K_S + \Theta_1. \qquad (S, \Theta_1) \text{ is lc}$$

$$(K_Y + (1-\eta)\Gamma + S)|_S = K_S + \Theta_2. \qquad K_S + \Theta_2 \equiv 0.$$

$$(K_Y + \Gamma + S)|_S = K_S + \Theta. \; , \qquad \text{coeff } \Theta \subseteq D(I).$$

$$(1-\varepsilon)\Theta \leq \Theta_1 \leq \Theta_2 \leq \Theta$$

$\Longrightarrow$ (S, Θ_2) is log canonical, $\quad \Theta_1 \leq \Theta_2$.

$\Longrightarrow$ (S, Θ_1) is log canonical.

$\Longrightarrow$ $\Theta = \Theta_2$

$\longrightarrow\longleftarrow$ $\square$

Lemma 6.2: $D_{n-1} + A_{n-1} \Longrightarrow B_n$.

Proof: $(X_i, \Delta_i) \in \mathfrak{D}$ $\qquad \mathrm{Vol}(\Delta_i) \longrightarrow \infty$.

$\Gamma_i \sim_{\mathbb{R}} \varepsilon_i \Delta_i$, $\quad \varepsilon_i \longrightarrow 0$.

$(X_i, \Gamma_i + (1-\varepsilon_i)\Delta_i)$ not klt. $\qquad K_{X_i} + \Gamma_i + (1-\varepsilon_i)\Delta_i \equiv 0$.

$$\Gamma_i + (1-\varepsilon_i)\Delta_i \geq (1-\varepsilon_i)\Delta_i \qquad \longrightarrow\longleftarrow.$$

$\square$

$$\mathrm{Vol}(\Delta) \leq \left(\frac{n}{\varepsilon}\right)^n \quad \text{where } \varepsilon \text{ is as in Lemma 6.1.}$$

Birational boundedness:

Lemma 7.1: (X,Δ) pair, X proj. D big $\mathbb{R}$-divisor.

If $\operatorname{vol}(D) > (2n)^n$ there exists $V \longrightarrow B$ covering X.
s.t if x & y are general in X, then we may find $b \in B$.

$0 \leq D_b \sim_{\mathbb{R}} D$ s.t $(X, \Delta + D_b)$ is lc but not klt at both x & y
and there exists a unique non-klt place of $(X, \Delta + D_b)$ with
center V_b containing x.

Lemma 7.2: Assume C_{n-1} + A_{n-1}. Fix p a positive integer.

$\mathcal{B}_1 = \{ (X,\Delta) \mid$ klt of dim n proj, $K_X + \Delta$ big, $\boxed{p}\Delta$ integral or Δ standard $\}$
Then $\phi_{m(K_X+\Delta)}$ is birational for every $(X,\Delta) \in \mathcal{B}_1$.

Proof: Assume $K_X+\Delta$ ample.

Fix k s.t. $\mathrm{vol}(\boxed{k}(K_X+\Delta)) > (2n)^n$

Apply 7.1 to $k(K_X+\Delta)$ to get a family $V \longrightarrow B$,

$\nu: W \longrightarrow V_b$ normalization

$(K_X+\Delta+\Delta_b)|_W - (K_W+\Theta)$ pseff.

Θ has coefficients in some DCC set. $\longleftarrow$ A_{n-1}

$U \longrightarrow W$ log resolution, $K_U+\Psi \geq K_X+\Delta|_U$

so $K_U+\Psi$ is big.

$C_{n-1} \longrightarrow$ $\phi_{\boxed{\ell}(K_U+\Psi)}$ is birational where ℓ is fixed.

(3.5.5) $\Rightarrow$ $\phi_{km_0(K_X+\Delta)}$ is birational

$\mathrm{Vol}(K_X+\Delta) \geq 1 \Longrightarrow \mathrm{Vol}(2(n+1)(K_X+\Delta)) > (2n)^n$ $\quad k = 2(n+1)$.

$\mathrm{vol}(K_X+\Delta) < 1$.

$$(2n)^n < \mathrm{Vol}(k(K_X+\Delta)) \leq (4n)^n$$

Hence, $\mathrm{vol}(m_0 k(K_X+\Delta)) \leq (4m_0 n)^n$.

DCC of volumes $\mathrm{vol}(\alpha(K_X+\Delta)) > (2n)^n$ for $\alpha = \frac{2n}{\delta}$

where δ is the minimum volume. $\square$.

Lemma 7.3: $\mathcal{B} = \left\{ (X,\Delta) \;\middle|\; \begin{array}{l} X \text{ proj of dim } n, \ (X,\Delta) \text{ lc} \\ K_X+\Delta \text{ big } \& \ \operatorname{coeff}(\Delta) \subseteq I \end{array} \right\}$.

Assume C_{n-1} & A_{n-1}

There exists $\beta < 1$ s.t. if $(X,\Delta) \in \mathcal{B}$ then the pseff thresholds

$$\lambda = \inf \{ t \in \mathbb{R} \mid K_X + t\Delta \text{ is big} \}$$

is at most β.

Proof: Assume (X,Δ) is snc.

Assume $\lambda > \frac{1}{2}$ so K_X is not pseff.

$0 \le D \sim_{\mathbb{R}} (K_X+\Delta)$. If $\varepsilon \ge 0$, then

$$(1+\varepsilon)(K_X + \lambda\Delta) \sim_{\mathbb{R}} K_X + \mu\Delta + \varepsilon D \quad \text{for some } \mu < \lambda.$$

klt for ε small enough

$(K_X+\lambda\Delta)$-MMP $X \overset{f}{\dashrightarrow} Y$, $K_Y + \Gamma$ is nef $f_*\lambda\Delta = \Gamma$.

Run $(K_Y + \mu f_*\Delta)$-MMP to get to a MFS $Y \longrightarrow Z$.

Assume $K_Y + \Gamma \equiv 0$.

$(X_\ell, \Delta_\ell) \in \mathcal{B}$ pseff thresholds are $\lambda_1 < \lambda_2 < \dots$

Let $J = \{\lambda_\ell\, i \mid i \in I,\ \ell \in \mathbb{N}\}$ J satisfies DCC.

B_n $\left\{\begin{array}{l} \mathrm{vol}(\Gamma) < C \text{ for any } (Y, \Gamma) \quad \mathrm{coeff}(\Gamma) \subseteq J. \\ \text{If } \alpha \text{ is the smallest element in } J, \quad G = \text{sum of comp of } \Gamma. \\ \text{Then } \mathrm{vol}(K_Y + G) \leq \frac{C}{\alpha^n}. \end{array}\right.$

We want to apply 7.2:

$K_X + \Delta$ is big.

D be the sum of comp of Δ. $K_X + D$ big. $D \geq \Delta$

$$K_X + D = f^*(K_Y + G) + F, \quad \text{where } F \text{ eff } \& \ f\text{-exc.}$$

Then, $\mathrm{Vol}(K_X + D) \leq \mathrm{Vol}(K_Y + G) \leq \frac{C}{\alpha^n}$.

(X_ℓ, D_ℓ), you may pick $r \in \mathbb{N}$: $K_{X_\ell} + \Theta_\ell := K_{X_\ell} + \frac{r-1}{r} D_\ell$ is big

(7.2) $\Longrightarrow$ (X_ℓ, Θ_ℓ) are birationally bounded

$\Longrightarrow$ (X_ℓ, Δ_ℓ) is birationally bounded.

Hence, $\mathrm{vol}(K_{X_\ell} + \Delta_\ell) \geq \delta$ for every ℓ.

$$\delta \leq \mathrm{vol}(K_X + \Delta) \leq \mathrm{vol}\left(K_Y + \frac{1}{\lambda}\Gamma\right) = \left(\frac{1}{\lambda} - 1\right)^n \mathrm{vol}(Y, \Gamma) \leq \left(\frac{1}{\lambda} - 1\right)^n C. \quad \square$$

Birational boundedness:

Lemma 7.4: $C_{n-1} + A_{n-1} + B_n \Longrightarrow C_n$.

Proof: α is the smallest element in I.

Assume (X, Δ) is log smooth and klt.

pseff threshold of Δ is at most $\beta < 1$.

Pick p: $\quad p > \dfrac{2}{\alpha(1-\beta)}$.

Observe: $\quad \dfrac{\lfloor p\alpha \rfloor}{p} > \dfrac{\alpha(1+\beta)}{2}$.

controlled denominators.

Hence: $\quad \dfrac{\beta+1}{2} \Delta \leq \Delta_{\lfloor p \rfloor} \leq \Delta$.

$K_X + \Delta_{\lfloor p \rfloor}$ is big.

$\phi_{m(K_X + \Delta_{\lfloor p \rfloor})}$ is birational by 7.2

$\Downarrow$

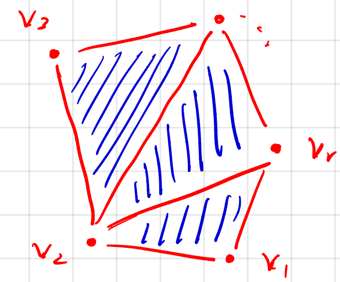
$\phi_{m(K_X + \Delta)}$ is also birational. $\square$.

Regularity and log canonical thresholds:

(X,Δ) lc, $\quad \mathrm{reg}(X,\Delta) := \dim \mathcal{D}(X,\Delta)$.

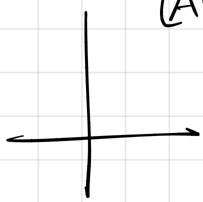
$\mathcal{D}(X,\Delta)$?

$$(Y, S_1 + \dots + S_r + \Delta_Y)$$
$$\Big\downarrow \text{ dlt mod.}$$
$$(X,\Delta)$$



reg 1

$(\mathbb{A}^2, L_1 + L_2)$



$\mathcal{D} =$

$\mathcal{D} := \bullet$

ACC for lct's of the same dimension holds.

thresholds of toric divisors on toric varieties $= 1$.

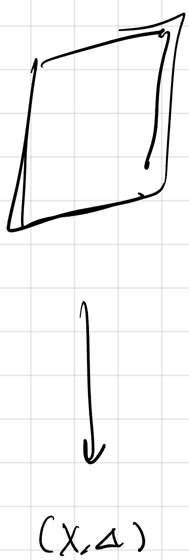
$$\mathrm{LCT}(c; I) = \left\{ \mathrm{lct}((X,\Delta); D) \;\middle|\; \begin{array}{l} (X,\Delta) \text{ has dim } n \\ (X,\Delta+tD) \text{ is lc of reg } r \\ c = n - r \end{array} \right\}$$

satisfies the ACC.

threshold.

Global-to-local:

S   S projective



global ACC.

dim − reg is small.

we expect to find many divisors with coeff 1 intersecting each other